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Factorization property of generalized s-selfdecomposable measures and class L^f distributions[1]

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Abstract. The method of *random integral representation*, that is, the method of representing a given probability measure as the probability distribution of some random integral, was quite successful in the past few decades. In this note we will find such a representation for generalized s-selfdecomposable and selfdecomposable distributions that have the *factorization property.* These are the classes $\mathcal{U}^f_\beta$ and L^f, respectively

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In probability theory, from its very beginning, characteristic functions (Fourier transforms) were used to describe measures and to prove limiting distributions theorems. In the past few decades many classes of probability measures (e.g. selfdecomposable measures , n-times selfdecomposable, s-selfdecomposable, type G distribution, etc.) were characterized in terms of

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distributions of some random integrals; cf. Jurek (1985, 1988) , Jurek and Vervaat (1983), Jurek and Mason (1993), Jurek and Yor (2004), Iksanov, Jurek and Schreiber (2004) and recently Aoyama and Maejima (2007). More precisely, for each of those classes one integrates a fixed deterministic function with respect to a class of Lévy processes, with possibly a time scale change.

Moreover, what we must emphasize here is that from the random integral representations easily follow those in terms of characteristic functions, and also one can infer from them new convolution factorizations or decompositions. Thus the random integral representations provide a new method in the area called *the arithmetic of probability measures*; cf. Cuppens (1975) or Linnik and Ostrovskii (1977).

In this note we consider more specific situations. Namely, for a convolution semigroup $\mathcal{C}$ of distributions of some random integrals and a measure $\mu \in \mathcal{C}$ we are interested in decompositions of the form

$$\mu = \mu_1 * \rho, \quad \mu_1 \in \mathcal{C}, \tag{1}$$

for some probability measure ρ that is intimately related to the measure μ_1.

This paper was inspired by questions related to the class L^f of selfdecomposable measures having the so called *factorization property* that was introduced and investigated in Iksanov, Jurek and Schreiber (2004).

Finally, let us note that the random integral representations for classes $\mathcal{U}_\beta^f$ (Corollary 1(a)) and L^f (Corollary 3) provide more examples for the conjectured "meta-theorem" in *The Conjecture* on www.math.uni.wroc.p/$\sim$zjjurek or see Jurek (1985) and (1988).

1. Notation and the results. Our results are presented for probability measures on Euclidean space $\mathbb{R}^d$. However, our proofs are such that they hold true for measures on infinite dimensional real separable Banach space E with the *scalar product* replaced by the *bilinear form* between $E' \times E$ and $\mathbb{R}$; E' denotes the topological dual of E and, of course, $(\mathbb{R}^d)' = \mathbb{R}^d$; cf. Araujo-Giné (1980), Chapter III. In particular, one needs to keep in mind Remark 1, below.

Let ID and $ID_{\log}$ denote all infinitely divisible probability measures (on $\mathbb{R}^d$ or E) and those that integrate the logarithmic function $\log(1 + ||x||)$, respectively. Let $Y_\nu(t), t \geq 0$ denote an $\mathbb{R}^d$ (or E) - valued Lévy process, i.e., a process with stationary independent increments, starting from zero, and with paths that continuous from the right and with finite left limits, such that ν is its probability distribution at time 1: $\mathcal{L}(Y_\nu(1)) = \nu$, where ν can be any ID probability measure.

Throughout the paper $\mathcal{L}(X)$ will denote the probability distribution of an $\mathbb{R}^d$-valued random vector (or a Banach space E-valued random elements if the Reader is interested in that generality).

Definition 1. *For $\beta > 0$ and a Lévy process Y_ν, let us define*

$$\mathcal{J}^\beta(\nu) := \mathcal{L}\big(\int_0^1 t^{1/\beta}\, dY_\nu(t)\big) = \mathcal{L}\big(\int_0^1 t\, dY_\nu(t^\beta)\big), \quad \mathcal{U}_\beta := \mathcal{J}^\beta(ID). \tag{2}$$

To the distributions from $\mathcal{U}_\beta$ we refer to as generalized s-selfdecomposable distributions.

The classes $\mathcal{U}_\beta$ were already introduced in Jurek (1988) as the limiting distributions in some schemes of summing independent variables. The terminology has its origin in the fact that distributions from the class $\mathcal{U}_1 \equiv \mathcal{U}$ were called *s-selfdecomposable distribution* (the "*s*-", stands here for *the shrinking operations* that were used originally in the definition of $\mathcal{U}$); cf. Jurek (1985), (1988) and references therein.

Proposition 1. ***A factorization of generalized s-selfdecomposable distribution.*** *In order that a generalized s-selfdecomposable distribution $\mu = \mathcal{J}^\beta(\rho)$, from the class $\mathcal{U}_\beta$, convoluted with its background measure ρ is again in the class $\mathcal{U}_\beta$ it is sufficient and necessary that $\rho \in \mathcal{U}_{2\beta}$.*

More explicitly,

$$[\,\mathcal{J}^\beta(\rho) * \rho = \mathcal{J}^\beta(\nu)\,] \Longleftrightarrow [\,\rho = \mathcal{J}^{2\beta}(\nu^{*\frac{1}{2}})\,] \tag{3}$$

Furthermore, for each $\tilde{\mu} \in \mathcal{U}_\beta$ there exists a unique $\tilde{\rho} \in \mathcal{U}_{2\beta}$ such that $\tilde{\mu} = \mathcal{J}^\beta(\tilde{\rho}) * \tilde{\rho}$ and $\mathcal{J}^{2\beta}\big(\tilde{\mu}\big) = \mathcal{J}^\beta\big((\tilde{\rho})^{*2}\big)$

Let us denote by $\mathcal{U}^f_\beta$ the class of generalized s-selfdecomposable admitting *the factorization property*, i.e, $\mu := \mathcal{J}^\beta(\rho) \in \mathcal{U}_\beta$ has the factorization property if $\mathcal{J}^\beta(\rho) * \rho \in \mathcal{U}_\beta$.

Corollary 1. *For $\beta > 0$ we have equalities*

$$(a)\ \ \mathcal{U}^f_\beta = \mathcal{J}^{2\beta}(\mathcal{U}_\beta) = \mathcal{J}^{2\beta}(\mathcal{J}^\beta(ID)) = \\ = \{\mathcal{L}(\int_0^1 (1-\sqrt{t})^{1/\beta}\, dY_\nu(t)) : \nu \in ID\}.$$

(b) $\mathcal{U}_\beta = \{\mathcal{J}^\beta(\rho) * \rho : \rho \in \mathcal{U}_{2\beta}\}$.

Taking in Proposition 1 $\beta = 1$ we get the following

Corollary 2. ***Factorization of s-selfdecomposable distributions.*** *An s-selfdecomposable distribution $\mu = \mathcal{J}(\rho)$ convoluted with ρ is again s-selfdecomposbale if and only if $\rho \in \mathcal{U}_2$. Thus we have $\mathcal{U}^f = \mathcal{J}^2(\mathcal{U})$.*

More explicitly

$$[\,\mathcal{J}(\rho) * \rho = \mathcal{J}(\nu)\,] \Longleftrightarrow [\,\rho = \mathcal{J}^2\big(\nu^{*\frac{1}{2}}\big)\,]. \qquad (4)$$

Moreover, for each $\tilde{\mu} \in \mathcal{U}$ there exist a unique $\rho \in \mathcal{U}_2$ such that $\tilde{\mu} = \mathcal{J}(\tilde{\rho}) * \tilde{\rho}$ and $\mathcal{J}^2\big(\tilde{\mu}\big) = \mathcal{J}\big((\tilde{\rho})^{*2}\big)$. Consequently, $\mathcal{U} = \{\mathcal{J}^2(\rho) * \rho : \rho \in \mathcal{U}\}$.

Following Jurek-Vervaat (1983) or Jurek (1985) we recall the following

Definition 2. *For a measure $\nu \in ID_{\log}$ and a Lévy process Y_ν let us define*

$$\mathcal{I}(\nu) := \mathcal{L}\Big(\int_0^\infty e^{-s}\, dY_\nu(s)\Big), \quad L := \mathcal{I}(ID_{\log}) \qquad (5)$$

and distributions from L are called selfdecomposable or Lévy class L distributions.

In classical probability theory the selfdecomposability (or in other words, the Lévy class L distributions) is usually defined via some decomposability property or by scheme of limiting distributions. However, since Jurek-Vervaat (1983) we know that the class L coincides with the class of distributions of random integrals given in (5) and thus it is used in this note as its definition.

Before going further, let us recall the following example that led to, and justified interest in, that kind of investigations/factorizations.

Example. For two dimensional Brownian motion $\mathbf{B}_t := (B_t^1, B_t^2)$, the process

$$\mathcal{A}_t := \int_0^t B_s^1\, dB_s^2 - B_s^2\, dB_s^1, \quad t > 0,$$

called *Lévy's stochastic area integral*, admits the following factorization

$$\chi(t) := E[e^{it\mathcal{A}_u} | \mathbf{B}_u = (\sqrt{u}, \sqrt{u})] = \frac{tu}{\sinh tu} \cdot \exp[-(tu\, \cosh tu - 1)], \qquad (6)$$

cf. P. Lévy (1951) or Yor (1992), p. 19.

Iksanov-Jurek-Schreiber (2004), p. 1367, proved that the factorization (6) may be interpreted as follows: if ν is the probability measure with the characteristic function $t \to \exp[-(tu\, \cosh tu - 1)]$ then $\mathcal{I}(\nu)$ has the characteristic function $t \to \frac{tu}{\sinh tu}$, and also

$$\mathcal{I}(\nu) * \nu = \mathcal{I}(\rho), \quad \text{for some} \quad \rho \in ID_{\log}; \qquad (7)$$

i.e., $\mathcal{I}(\nu)$ is selfdecomposable and when convoluted with its background driving probability measure ν we again get a distribution from the class L.

Let us note that the convolution factorizations (7), (3) and (4) are of the form described in (1), with different semigroups $\mathcal{C}$.

Proposition 2. ***Random integral representation of*** $\boldsymbol{\mathcal{I}(\mathcal{J}^{\beta}(ID_{\log}))}$.
For $\nu \in ID_{\log}$ *and* $\beta > 0$

$$\mathcal{I}(\mathcal{J}^{\beta}(\nu)) = \mathcal{L}\Big(\int_0^{\infty} e^{-s}\, dY_{\nu}(\sigma_{\beta}(s))\Big), \tag{8}$$

where $Y_{\nu}(t), t \geq 0$ *is a Lévy process such that* $\mathcal{L}(Y_{\nu}(1)) = \nu$ *and the deterministic inner clock* σ_{β} *is given by* $\sigma_{\beta}(s) := s + \frac{1}{\beta}e^{-\beta s} - \frac{1}{\beta},\ s \geq 0.$

From Proposition 1 (ii) in Iksanov-Jurek-Schreiber (2004) and taking $\beta = 1$ in Proposition 2 we get

Corollary 3. *For the class,* L^f*, of selfdecomposable distributions with factorization property, we have the following random integral representation*

$$L^f = \Big\{\mathcal{L}\Big(\int_0^{\infty} e^{-s}\, dY_{\nu}(s + e^{-s} - 1)\Big)\ :\ \nu \in ID_{\log}\Big\}. \tag{9}$$

2. Proofs. For a probability Borel measures μ on $\mathbb{R}^d$, its *characteristic function* $\hat{\mu}$ is defined as

$$\hat{\mu}(y) := \int_{\mathbb{R}^d} e^{i<y,x>}\mu(dx),\ y \in \mathbb{R}^d,$$

where $< \cdot, \cdot >$ denotes the scalar product; (in case one wants to have results on Banach spaces $< \cdot, \cdot >$ is the bilinear form on $E' \times E$ and $y \in E'$).
Recall that for infinitely divisible measures μ their characteristic functions admit the following Lévy-Khintchine formula

$$\hat{\mu}(y) = e^{\Phi(y)},\ y \in \mathbb{R}^d, \quad \text{and the exponents } \Phi \text{ are of the form}$$

$$\Phi(y) = i < y, a > -\frac{1}{2} < y, Sy > + \int_{\mathbb{R}^d \setminus \{0\}} [e^{i<y,x>} - 1 - i < y, x > 1_B(x)]M(dx), \tag{10}$$

where a is a *shift vector*, S is a *covariance operator* corresponding to the Gaussian part of μ and M is a *Lévy spectral measure.* Since there is a one-to-one correspondence between a measure $\mu \in ID$ and the triples a, S and

M in its Lévy-Khintchine formula (10) we will write $\mu = [a, S, M]$. Finally, let recall that

$$M \text{ is Lévy spetral measure on } \mathbb{R}^d \text{ iff } \int_{\mathbb{R}^d} \min(1, ||x||^2) M(dx) < \infty \quad (11)$$

(For infinite divisibility of probability measures on Banach spaces we refer to the monograph by Araujo-Giné (1980), Chapter 3, Section 6, p. 136. Let us stress that the characterization (11), of Lévy spectral measures, is in general NOT true in infinite dimensional Banach spaces ! However, it holds true in Hilbert spaces; cf. Parthasarathy (1967), Chapter VI, Theorem 4.10.)

Before proving Proposition 1, let us note the following auxiliary facts.

Lemma 1. *(a) For the mapping $\mathcal{J}^\beta$ and $\nu \in ID$ we have*

$$\widehat{\mathcal{J}^\beta(\nu)}(y) = \exp \int_0^1 \log \widehat{\nu}(t^{1/\beta} y)\, dt = \exp \mathbf{E}[\log \widehat{\nu}(U^{1/\beta} y)], \quad y \in \mathbb{R}^d \ (or\ E'). \tag{12}$$

and U is a random variable uniformly distributed over the unit interval $(0, 1)$.

(b) The mapping $\mathcal{J}^\beta$ is one-to-one. More explicitly we have that

$$\frac{d}{ds}[s \log \widehat{\mathcal{J}^\beta(\nu)}(s^{1/\beta} y)]|_{s=1} = \log \hat{\nu}(y), \quad \textit{for all } y \in \mathbb{R}^d \ (or\ E'). \tag{13}$$

(c) The mappings $\mathcal{J}^\beta, \beta > 0$ commute, i.e., for $\beta_1, \beta_2 > 0$ and $\nu \in ID$, $\mathcal{J}^{\beta_1}(\mathcal{J}^{\beta_2}(\nu)) = \mathcal{J}^{\beta_2}(\mathcal{J}^{\beta_1}(\nu))$.

(d) For probability measures ν_1, ν_2 and $c > 0$ we have that

$$\mathcal{J}^\beta(\nu_1 * \nu_2) = \mathcal{J}^\beta(\nu_1) * \mathcal{J}^\beta(\nu_2); \quad (\mathcal{J}^\beta(\nu))^{*c} = \mathcal{J}^\beta(\nu^{*c}) \tag{14}$$

(e) For $\beta > 0$ and $\rho \in ID$ we have the identity

$$\mathcal{J}^{2\beta}(\mathcal{J}^\beta(\rho) * \rho) = \mathcal{J}^\beta(\rho^{*2}) \tag{15}$$

Proof of Lemma 1. Part (a) follows from the definition of the random integrals and is a particular form (take matrix $Q = I$) of Theorem 1.3 (a) in Jurek (1988).

For the claim (b) note that for each fixed y we have

$$\log \widehat{\mathcal{J}^\beta(\nu)}(s^{1/\beta} y) = s^{-1} \int_0^s \log \hat{\nu}(r^{1/\beta} y) dr, \quad s \in \mathbb{R}^+.$$

This gives the formula in (b), similarly as in Jurek (1988), p. 484. Equalities in (c) and (d) are also consequences of (a); cf. Jurek(1988), Theorem 1.3 (a) and (c).

Finally, for the identity in (e) note, using (14) that

$$\log \Big(\mathcal{J}^{2\beta}\big(\mathcal{J}^{\beta}(\rho)*\rho\big)\Big)\widehat{\ }(y) = \int_0^1 \log \big(\mathcal{J}^{\beta}(\rho)*\rho\big)\Big)\widehat{\ }(s^{1/2\beta}y) =$$

$$\int_0^1\int_0^1 \log\hat{\rho}(t^{1/\beta}s^{1/2\beta}y)dt\,ds + \int_0^1 \log\hat{\rho}(s^{1/2\beta}y)ds \qquad (\text{put } t^2 s =: u)$$

$$= \int_0^1 1/2 \int_0^s \log\hat{\rho}(u^{1/2\beta}y)(us)^{-1/2}du\,ds + \int_0^1 \log\hat{\rho}(s^{1/2\beta}y)ds$$

$$= \int_0^1 \log\hat{\rho}(u^{1/2\beta}y)\,u^{-1/2}\big(1/2\int_u^1 s^{-1/2}ds\big)\,du + \int_0^1 \log\hat{\rho}(s^{1/2\beta}y)ds =$$

$$\int_0^1 u^{-1/2}\log\hat{\rho}(u^{1/2\beta}y)du = 2\int_0^1 \log\hat{\rho}(u^{1/2\beta}y)d(u^{1/2}) =$$

$$\int_0^1 \log\hat{\rho^{*2}}(s^{1/\beta}y)ds = \log\big(\mathcal{J}^{\beta}(\rho^{*2})\big)\widehat{\ }(y), \tag{16}$$

which completes the proof of Lemma 1.

Proof of Proposition 1. Suppose we have that $\mathcal{J}^{\beta}(\rho)*\rho = \mathcal{J}^{\beta}(\nu)$. Then by the properties described in Lemma 1,

$$\mathcal{J}^{\beta}\big(\mathcal{J}^{2\beta}(\nu)\big) = \mathcal{J}^{2\beta}\big(\mathcal{J}^{\beta}(\nu)\big) = \mathcal{J}^{2\beta}\big(\mathcal{J}^{\beta}(\rho)*\rho\big) = \mathcal{J}^{\beta}(\rho^{*2}),$$

and hence $\rho^{*2} = \mathcal{J}^{2\beta}(\nu)$, i.e., $\rho = (\mathcal{J}^{2\beta}(\nu))^{*1/2} = \mathcal{J}^{2\beta}(\nu^{*1/2})$, which proves the necessity. The converse claim also follows from the above reasoning.

For the last part, let us note that if $\tilde{\mu} = \mathcal{J}^{\beta}(\nu) \in \mathcal{U}_{\beta}$ then taking $\rho := \mathcal{J}^{2\beta}(\nu^{*1/2}) \in \mathcal{U}_{2\beta}$ one gets the required equality.

Proof of Corollary 1. Note that $\nu = \mathcal{J}^{\beta} \in \mathcal{U}_{\beta}^{f}$ iff $\mathcal{J}^{\beta}(\rho)*\rho \in \mathcal{U}_{\beta}$ iff $\rho \in \mathcal{U}_{2\beta}$, by (3) in Proposition 1. Last equality is from the Example **(a)** from Czyżewska-Jankowska and Jurek (2008). Similarly one gets part (b) using Proposition 1 and Lemma 1 (e).

Proposition 1 can be expressed in terms of characteristic functions as follows:

Corollary 4. *In order that*

$$\exp\int_0^1 \log\hat{\rho}\,\big(t^{1/\beta}y\big)dt\ \cdot\ \hat{\rho}\,(y) = \exp\int_0^1 \log\hat{\nu}\,\big(t^{1/\beta}y\big)dt, \qquad y\in\mathbb{R}^d\ (or\ E')$$

for some μ and ρ in ID it is necessary and sufficient that

$$\hat{\rho}\,(y) = \exp\int_0^1 \tfrac{1}{2}\log\hat{\nu}\,\big(t^{1/(2\beta)}y\big)dt;$$

or in terms of the Lévy spectral measures as:

Corollary 5. *In order to have the equality*

$$\int_0^1 M(t^{-1/\beta}A)\,dt + M(A) = \int_0^1 G(t^{-1/\beta}A)\,dt, \qquad \textit{for each Borel } A \in \mathcal{B}_0,$$

for some Lévy spectral measures M and G, it is necessary and sufficient that

$$M(A) = \int_0^1 \tfrac{1}{2} G(t^{-1/(2\beta)}A)\,dt, \qquad \textit{for each} \;\; A \in \mathcal{B}_0,$$

because if $\rho = [a, S, M]$ then the left hand side in the Corollary is the Lévy spectral measure of $\mathcal{J}^\beta(\rho) * \rho$.

For references let state the following

Lemma 2. *(i) If $\nu = [a, R, M]$ and $\mathcal{J}^\beta(\nu) = [a^{(\beta)}, R^{(\beta)}, M^{(\beta)}]$ then*

$$a^{(\beta)} := \tfrac{\beta}{(1+\beta)}\, a + \int_0^1 t^{1/\beta} \int_{\{1<\|x\|\le t^{-1/\beta}\}} x\; M(dx)\; dt$$
$$= \frac{\beta}{\beta+1}\,(a + \int_{(||x||>1)} x\,||x||^{-1-\beta} M(dx)\,); \qquad R^{(\beta)} := \tfrac{\beta}{2+\beta}\, R;$$
$$M^{(\beta)}(A) := \int_0^1 T_{t^{1/\beta}}\; M(A)\; dt, \;\; \textit{for each } A \in \mathcal{B}_0.$$

(ii) For $\beta > 0$, we have that $\mathcal{J}^\beta(\nu) \in ID_{\log}$ if and only if $\nu \in ID_{\log}$.

Proof of Lemma 2. (i) Uniqueness of the triplets: *a shift vector a , Gaussian covariance R and Lévy spectral measure M* in the Lévy-Khintchine formula and equation (12) in Lemma 1 give the expressions for $a^{(\beta)}, R^{(\beta)}$ and for $M^{(\beta)}$; for details cf. formulas (1.10), (1.11) and (1.12) in Jurek (1988), with the matrix $Q = I$.

For part (ii), note that since we have

$$\int_{\{\|x\|>1\}} \log \|x\|\; M^{(\beta)}(dx) = \int_0^1 \int_{\{\|x\|>1\}} \log \|x\|\; T_{t^{1/\beta}} M(dx)\; dt =$$
$$= \int_0^1 \int_{\{\|t^{1/\beta}x\|>1\}} \log \|t^{1/\beta}x\|\; M(dx)\,dt = \int_0^1 \int_{\{\|x\|>\frac{1}{t^{1/\beta}}\}} \log\,(t^{1/\beta}\|x\|)\; M(dx)\,dt =$$

$$= \int\limits_{\{\|x\|>1\}} \int\limits_{\|x\|^{-1/\beta}}^{1} \log\left(t^{1/\beta}\|x\|\right) dt\, M(dx) = \int\limits_{\{\|x\|>1\}} \frac{1}{\|x\|^{\beta}} \int\limits_{\|x\|^{1-1/\beta^2}}^{\|x\|} \beta w^{\beta-1} \log w\, dw\, M(dx) =$$

$$= \int\limits_{\{\|x\|>1\}} \frac{1}{\|x\|^{\beta}} \left[w^{\beta}\log w - \frac{1}{\beta} w^{\beta} \Big|_{w=\|x\|^{1-1/\beta^2}}^{w=\|x\|} \right] M(dx) =$$

$$= \int\limits_{\{\|x\|>1\}} \log\|x\|\, M(dx) - \int\limits_{\{\|x\|>1\}} [\tfrac{1}{\beta} + \tfrac{1}{\|x\|^{1/\beta}}\big((1-\tfrac{1}{\beta^2})\log\|x\| - \tfrac{1}{\beta}\big)]\, M(dx)$$

and the last integral is finite (the integrand function is bounded on $(||x|| > 1)$ and Lévy spectral measures M are finite on the complements of all neighborhoods of zero; comp. (11)), therefore from the above we conclude that

$$[\int\limits_{\{\|x\|>1\}} \log\|x\|\, M^{(\beta)}(dx) < \infty] \ \text{ iff } \ [\int\limits_{\{\|x\|>1\}} \log\|x\|\, M(dx) < \infty].$$

But since the function $u \to \log(1+u)$, for $u > 0$, is sub-additive therefore we may apply Proposition 1.8.13 in Jurek-Mason (1993) and infer the claim (ii). This completes the proof of Lemma 2.

Proof of Proposition 2. If $\nu \in ID_{\log}$ then, by Lemma 2, $\mathcal{J}^{\beta}(\mu) \in ID_{\log}$ and thus the improper random integral $\int_0^\infty e^{-s} dY_{\mathcal{J}^{\beta}(\nu)}(s)$ converges (is well-defined) almost surely (in probability and in distribution); cf. Jurek-Vervaat (1983), Lemma 1.1 or Jurek (1985). Hence and Lemma 1(a) we get that

$$\log\left(\mathcal{I}\left(\mathcal{J}^{\beta}(\nu)\right)\right)\widehat{\ }(y) = \int_0^\infty \log \widehat{\mathcal{J}^{\beta}(\nu)}(e^{-s}y)ds = \int_0^\infty \int_0^1 \log\hat{\nu}(v^{1/\beta}e^{-s}y)dvds$$

$$= \int_0^1 \int_0^{v^{1/\beta}} \log\hat{\nu}(uy)u^{-1}du = \int_0^1 (\int_{u^{\beta}}^1 dv) \log\hat{\nu}(uy)u^{-1}dudv =$$

$$\int_0^1 \log\hat{\nu}\,(uy)\big(u^{-1} - u^{\beta-1}\big)\, du = \int_0^\infty \log\hat{\nu}\big(e^{-s}y\big)\big(1 - e^{-\beta s}\big)\, ds =$$

$$= \int_0^\infty \log\hat{\nu}\big(e^{-s}y\big)\, d\sigma_{\beta}(s).$$

On the other hand, the random integral

$$\int_0^\infty e^{-s}\, dY_{\nu}(\sigma_{\beta}(s)) := \lim_{b\to\infty} \int_0^b e^{-s}\, dY_{\nu}(\sigma_{\beta}(s)) \text{ exists in distribution,}$$

(or in probability or almost surely) because the function

$$y \to \lim_{b\to\infty} \Big(\mathcal{L}\big(\int_0^b e^{-s}\, dY_\nu(\sigma_\beta(s))\big)\Big)\widehat{}\,(y)$$
$$= \lim_{b\to\infty} \exp \int_0^b \log \hat{\nu}(e^{-s}y)\, d\sigma_\beta(s) = \exp \int_0^\infty \log \hat{\nu}(e^{-s}y)\, d\sigma_\beta(s),$$

is a characteristic function. Moreover, we have that

$$\mathcal{I}\big(\mathcal{J}^\beta(\nu)\big) = \mathcal{L}\big(\int_0^\infty e^{-s}\, dY_\nu\big(\sigma_\beta(s)\big)\big),$$

which completes a proof of Proposition 2.

Remark 1. *Our argument above is valid for infinite dimensional Banach spaces, although one should be aware that in that generality convergence of characteristic functions to a characteristic function does not guarantee weak convergence of corresponding distributions (probability measures); cf. Araujo-Gine (1980), Theorem 4.19 on p. 29.*

Proof of Corollary 3. Recall that by definition $L^f = \{\mathcal{I}(\mu) : \mathcal{I}(\mu) * \mu \in L\}$. However, in view of Proposition 1 (ii) in Iksanov-Jurek-Schreiber (2004) we have $L^f = \mathcal{I}(\mathcal{J}(ID_{\log})$. Consequently, taking $\beta = 1$ in Proposition 2 we get the corollary.

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